Lecture 1

Afshin Goodarzi*

Royal Institute of Technology, Department of Mathematics, S-100 44, Stockholm, Sweden.

Cell Complexes

A cell complex, roughly speaking, is a decomposition of a topological space into various dimensional blocks (cells) of the "same type". Our main objects of interest in these lectures are simplicial complexes. These are cell complexes that their building blocks are simple geometric objects (simplices) and the information on how to glue them back can be captured entirely in a simple combinatorial manner (via partially ordered sets). Thus simplicial complexes can be considered both as geometric and combinatorial objects. We start with the former and delay the latter for the later.

Geometric Simplicial Complexes

Affine Hull. Let $V = \{v_0, v_1, \ldots, v_d\}$ be a set of $d+1$ distinct points in $\mathbb{R}^N$. A point x in $\mathbb{R}^N$ is an *affine combination* of $v_0, v_1, \ldots, v_d$ if $x = \sum_{i=0}^{d} t_i v_i$ for some real numbers t_i that $\sum_{i=0}^{d} t_i = 1$. The *affine hull* of V is the set of all affine combinations of $v_0, v_1, \ldots, v_d$. We say that $v_0, v_1, \ldots, v_d$ are called *affinely independent* if the vectors $v_1 - v_0, \ldots, v_d - v_0$ are linearly independent in $\mathbb{R}^N$ or equivalently if the affine hull of V is a d-dimensional plane. In particular, if $v_0, v_1, \ldots, v_d$ are affinely independent in $\mathbb{R}^N$, then $d \leq N$.

Convex Hull. An affine combination $u = \sum_{i=0}^{d} t_i v_i$ is said to be *convex* if $t_i \geq 0$ for all i. The set of all convex combinations of elements in V

$$\operatorname{conv}(V) = \left\{ x \in \mathbb{R}^N \middle| \quad x = \sum_{i=0}^{d} t_i v_i, \quad \sum_{i=0}^{d} t_i = 1, \quad 0 \leq t_i \right\} \tag{1.1}$$

is called the *convex hull* of V.

Simplices. The convex hull σ of a set V of $d+1$ affinely independent points in $\mathbb{R}^N$ is called a d-dimensional *simplex* (or a d-simplex). In Figure 1 d-simplices for $d = 0, 1, 2, 3$ are shown. If V is empty, we consider $\operatorname{conv}(V) = \emptyset$ as (-1)-simplex. A *face* of σ is the convex hull of any non-empty subset of V. Thus a face of σ is also a simplex. If τ is a face of σ we write $\tau \leq \sigma$. If, in addition, we want to mention that τ is different from σ, we write $\tau < \sigma$. The union $\operatorname{bd}(\sigma)$ of all proper faces of σ is called the *boundary* of σ. A d-simplex is homeomorphic to the unit d-ball $\mathbb{B}^d = \{x \in \mathbb{R}^d \mid |x| \leq 1\}$ and its boundary is homeomorphic to $(d-1)$-sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}$.

Definition 1 (Geometric Simplicial Complex). A *geometric simplicial complex* $\mathcal{K}$ in $\mathbb{R}^N$ is a finite non-empty collection of simplices in $\mathbb{R}^N$ such that

(1) if $\sigma \in \mathcal{K}$ and $\tau \leq \sigma$, then $\tau \in \mathcal{K}$, and

(2) if σ and τ are members of $\mathcal{K}$, then $\sigma \cap \tau$ is a face of σ and τ both.

*af.goodarzi@gmail.com

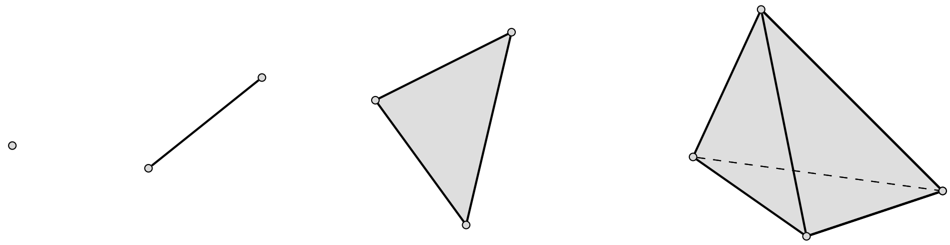

Figure 1: d-simplex for $d \in \{0, 1, 2, 3\}$

Faces. The members of $\mathcal{K}$ are called *faces* of $\mathcal{K}$. An inclusion-wise maximal face is a *facet*. A face of dimension d is called a *d-face*. A 0-face is called a *vertex* and a 1-edge is an *edge*. The set of vertices of $\mathcal{K}$ will be denoted by $V(\mathcal{K})$. The *dimension* $\dim \mathcal{K}$ of $\mathcal{K}$ is the maximum dimension of its faces. If $\mathcal{K}$ has dimension n and if $f_i = f_i(\mathcal{K})$ denotes the number of i-dimensional faces of $\mathcal{K}$, then the vector $f(\mathcal{K}) = (f_{-1}, f_0, f_1, \ldots, f_n)$ is called the *f-vector* of $\mathcal{K}$.

Subcomplexes. Let $\mathcal{L}$ be a non-empty subset of $\mathcal{K}$ so that for all $\sigma \in \mathcal{L}$ and all $\tau \leq \sigma$ one has $\tau \in \mathcal{L}$. Then $\mathcal{L}$ is a *subcomplex* of $\mathcal{K}$. A subcomplex is *full* if whenever it contains the vertices of a face σ of $\mathcal{K}$, it contains σ as well. Thus given $\mathcal{K}$ and a subset U of $V(\mathcal{K})$ there is a unique full subcomplex $\mathcal{L}$ with $U = V(\mathcal{L})$. In this situation we say that $\mathcal{L}$ is the subcomplex *induced* by U. The subcomplex $\mathcal{K}^{(n)}$ of all faces of dimension at most n in $\mathcal{K}$ is called the *n-skeleton* of $\mathcal{K}$.

Let σ be a face of $\mathcal{K}$. The *link* $\mathrm{lk}_\sigma(\mathcal{K})$ of σ in $\mathcal{K}$ is the subcomplex consisting of all faces τ such that (1) $\tau \cap \sigma = \emptyset$ and (2) there is a face in $\mathcal{K}$ that contain σ and τ both.

Triangulation. The *geometric realization* $||\mathcal{K}||$ of $\mathcal{K}$ is the union of all simplices in $\mathcal{K}$ equipped with the induced topology from $\mathbb{R}^N$. A *triangulation* of a topological space $\mathcal{X}$ is a geometric simplicial complex $\mathcal{K}$ together with a homeomorphism from $||\mathcal{K}||$ to $\mathcal{X}$. The simplicial complex in Figure 2 is triangulation of the 2-dimensional sphere with f-vector $(1, 6, 12, 8)$.

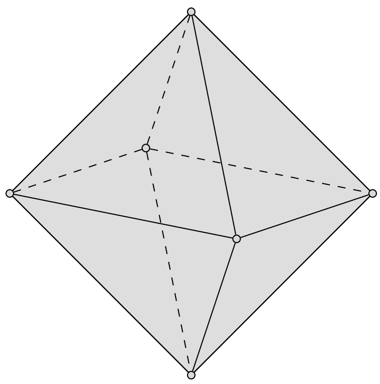

Figure 2: The boundary complex of Octahedron

Simplicial Maps. Let $\mathcal{L}$ and $\mathcal{K}$ be two geometric simplicial complexes and f be a function from $V(\mathcal{L})$ to $V(\mathcal{K})$ so that for any face $\sigma \in \mathcal{L}$ the image of vertices of σ span a face in $\mathcal{K}$. Then f can be extended to a continuous map $\varphi_f : ||\mathcal{L}|| \to ||\mathcal{K}||$. The map φ_f is called the *simplicial map* induced by f. The map φ_f is a *simplicial homeomorphism* if f is a bijection and a subset U of vertices of $\mathcal{L}$ spans a face in $\mathcal{L}$ if and only if $f(U)$ spans a face in $\mathcal{K}$. Alternatively, one can think of a simplicial map $\varphi : ||\mathcal{L}|| \to ||\mathcal{K}||$ as a continuous map such that for nay face σ of $\mathcal{L}$ the restriction $\varphi|_\sigma$ maps σ linearly onto a face of $\mathcal{K}$.

Subdivision. A subdivision of a simplicial complex $\mathcal{K}$ is a simplicial complex $\mathcal{K}'$ such that (1) any face of $\mathcal{K}'$ is contained in a face of $\mathcal{K}$ and (2) any face of $\mathcal{K}$ is a union of some faces of $\mathcal{K}'$. Since the union of all faces in $\mathcal{K}$ is the union of all faces in $\mathcal{K}$, one has $||\mathcal{K}'|| = ||\mathcal{K}||$.

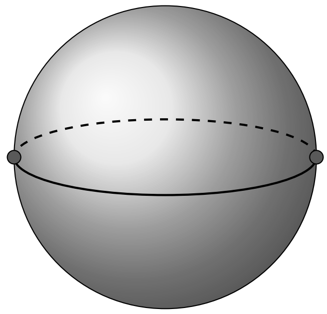

Figure 3: A regular cell decomposition of 2-sphere

CW-Complexes

In this part we introduce some more general cell decompositions. Although our main interest is in simplicial complexes, but considering more general classes of complexes has many advantages. For instance, it enables us to make use of operations that do not preserve simplicial complexes.

Definition 2. A (finite) *CW-complex* is a pair $(\mathcal{X}, \Sigma)$ where $\mathcal{X}$ is a Hausdorff topological space and Σ is a finite partition of $\mathcal{X}$ into open cells of various dimensions such that for each open d-cell $\sigma \in \Sigma$ there exists a continuous map $\varphi_\sigma : \mathbb{B}^d \to \mathcal{X}$, called the *characteristic map* of σ, that satisfies the following properties:

- $\mathring{\varphi}_\sigma : \mathring{\mathbb{B}}^d \to \mathcal{X}$ is a homeomorphism onto σ; and
- $\mathrm{bd}(\sigma) := \varphi_\sigma(\mathbb{S}^{d-1})$ is contained in the union of some open cells in Σ of dimension less than d.

A CW-complex is called *regular* (or a *regular cell complex*) if for all $\sigma \in \Sigma$ the characteristic map of σ is a homeomorphism to the *closure* $\bar{\sigma}$ of σ in $\mathcal{X}$ (i.e., the image $\varphi_\sigma(\mathbb{B}^d)$) and $\mathrm{bd}(\sigma)$ is equal to the union of some open cells in Σ.

Example (Polyhedral Complexes)**.** A *convex polytope* in $\mathbb{R}^N$ is the convex hull of a finitly many points in $\mathbb{R}^N$. A *proper face* of a polytope is the intersection of the polytope with a *supporting hyperplane* (Recall that a hyperplane H supports a polytope P at Q if $P \cap H = Q$ and P lies in one of the closed halfspaces bounded by H). The boundary of a polytope (the union of its proper faces) is a regular CW sphere whose open cells are the interiors of the faces.

Homotopy Type

Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces and φ, ψ be continuous maps from $\mathcal{X}$ to $\mathcal{Y}$. A *homotopy* between φ and ψ is a continuous map $\Phi : \mathcal{X} \times [0,1] \to \mathcal{Y}$ such that $\Phi(x,0) = \varphi(x)$ and $\Phi(x,1) = \psi(x)$ for all $x \in \mathcal{X}$. We say that φ and ψ are *homotopic* if there is a homotopy between them.

Definition 3 (Homotopy Equivalence)**.** Two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are *homotopy equivalent*, $\mathcal{X} \simeq \mathcal{Y}$, if there exist continuous maps $\varphi : \mathcal{X} \to \mathcal{Y}$ and $\psi : \mathcal{Y} \to \mathcal{X}$ such that $\psi \circ \varphi$ is homotopic to identity map on $\mathbf{id}_\mathcal{X}$ and $\varphi \circ \psi$ is homotopic to $\mathbf{id}_\mathcal{Y}$. Two spaces that are homotopy equivalent are said to have the same *homotopy type*. If $\mathcal{X}$ has the same homotopy type as a single point space we say that $\mathcal{X}$ is *contractible*.

Strong Deformation Retract. Let $\mathcal{Y}$ be a subspace of $\mathcal{X}$. A *strong deformation retraction* of $\mathcal{X}$ onto $\mathcal{Y}$ is a continuous map $\Phi : \mathcal{X} \times [0,1] \to \mathcal{X}$ such that $\Phi(x,0) = x$ and $\Phi(x,1) \in \mathcal{Y}$ for all $x \in \mathcal{X}$, and $\Phi(y,t) = y$ for all $y \in \mathcal{Y}$ and all $t \in [0,1]$. The map $\mathbf{r} : \mathcal{X} \to \mathcal{Y}$ defined by $\mathbf{r}(x) = \Phi(x,1)$ is a *retraction* of $\mathcal{X}$ to $\mathcal{Y}$ (i.e., each point of $\mathcal{Y}$ is a fixed point). We say that $\mathcal{Y}$ is a *strong deformation retract* of $\mathcal{X}$ if there exists a strong deformation retraction of $\mathcal{X}$ onto $\mathcal{Y}$.

Lemma 4. *If $\mathcal{Y}$ is a strong deformation retraction of $\mathcal{X}$, then $\mathcal{Y}$ has the same homotopy type as $\mathcal{X}$.*

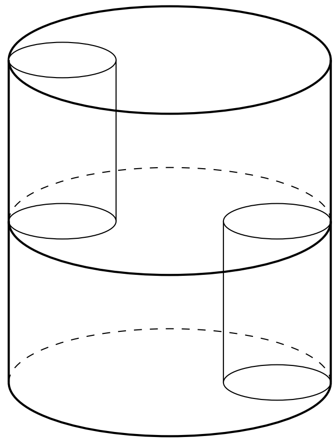

Figure 4: Bing's House

Collapsing. Suppose $\mathcal{K}$ is a regular cell complex and τ a $(d-1)$-face of it. We say that τ is a *free* $(d-1)$*-face* if there exists one and only one d-face σ of $\mathcal{K}$ that contain τ. Observe if $\mathcal{K}$ is a simplicial complex, then σ must be a facet of $\mathcal{K}$. Later we show that this is the case for any regular cell complex $\mathcal{K}$. Let $\mathcal{L}$ be the subcomplex of $\mathcal{K}$ obtain by removing τ and σ from $\mathcal{K}$. Then we write $\mathcal{K} \searrow^e \mathcal{L}$ and say $\mathcal{L}$ is obtained from $\mathcal{K}$ by an *elementary collapse*. We say that $\mathcal{K}$ *collapses* to $\mathcal{L}$, denoted by $\mathcal{K} \searrow \mathcal{L}$ if there is a sequence of regular cell complexes $\mathcal{K} = \mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_\ell = \mathcal{L}$ such that $\mathcal{K}_{i-1} \searrow^e \mathcal{K}_i$ for all $1 \le i \le \ell$. If $\mathcal{K}$ collapses to $\mathcal{L}$ we also say that $\mathcal{K}$ is obtained from $\mathcal{L}$ by *expansion* and we denote it by $\mathcal{L} \nearrow \mathcal{K}$. We say that $\mathcal{K}$ is *collapsible* if $\mathcal{K}$ collapses to one vertex.

Theorem 5. *If $\mathcal{K} \searrow \mathcal{L}$, then $\mathcal{K} \simeq \mathcal{L}$. In particular, a collapsible simplicial complex is contractible.*

The converse of Theorem 5 does not hold.

Example (Bing's House)**.** Start with the surface of a cylinder. Remove the interior of a tangent 2-disk from its roof and its ground floor. Add a horizental floor (a 2-disk with the interiors of two 2-disks removed) to separate the upper and the lower rooms. And finally add cylindral walls to separate the entrances from the rooms. Observe that in order to access the upper room one has to use the lower entrance and, similarly, in order to access the lower room one has to use the upper entrance. The Bing's house does not have any collapsible cell decomposition, since it cannot have any free edge. However, one can sees that the solid cylinder is an expansion of the Bing's house which implies the contractibility of this space.

Simple-Homotopy. Two regular cell complexes $\mathcal{K}$ and $\mathcal{L}$ have the same *simple-homotopy type* if there exists a finite sequence $\mathcal{K} = \mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_\ell = \mathcal{L}$ such that for each $1 \le i \le \ell$ either $\mathcal{K}_{i-1} \searrow \mathcal{K}_i$ or $\mathcal{K}_{i-1} \nearrow \mathcal{K}_i$.

It follows from Theorem 5 that if $\mathcal{K}$ and $\mathcal{L}$ have the same simple-homotopy type, then they have the same homotopy type. The converse is not true in general.

Theorem 6 (Whitehead)**.** *A regular cell complex $\mathcal{K}$ is contractible if and only if $\mathcal{K}$ has the simple-homotopy type of a vertex.*

References

[Coh73] Marshall M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York-Berlin, 1973.

[Mun75] James R. Munkres, *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

[Mun84] ______, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.